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ON EXACTLY m TIMES COVERS

BY

ZHI-WEI SUN*

Department of Mathematics
Nanjing University
Nanjing 210008
People's Republic of China

ABSTRACT

In this paper it is shown that if every integer is covered by $a_1+n_1\mathbb{Z},\ldots,a_k+n_k\mathbb{Z}$ exactly m times then for each $n=1,\ldots,m$ there exist at least $\binom{m}{n}$ subsets I of $\{1,\ldots k\}$ such that $\sum_{i\in I}1/n_i$ equals n. The bound $\binom{m}{n}$ is best possible.

Let $a+n\mathbb{Z}$ denote the arithmetic sequence $\{x\in\mathbb{Z}: x\equiv a \,(\operatorname{mod} n)\}$. We call

$$\{a_i+n_i\mathbb{Z}\}_{i=1}^k \tag{1}$$

an exactly m times cover if each integer belongs to exactly m of the sequences. An exactly one time cover is also said to be an exact cover.

It is well known [3] that $\sum_{i=1}^k 1/n_i$ equals m if (1) is an exactly m times cover, in particular $\sum_{i=1}^k 1/n_i$ equals 1 if (1) is an exact cover.

Porubský once asked whether each exactly m times cover is a union of m exact covers (cf. [2]). In 1976, Choi constructed an exactly 2 times cover which is not the union of two exact covers. (See Porubský [3].) For $m\ge 2$, the sequences in Choi's example, together with $m-2$ sequences $\mathbb{Z}$, form an exactly m times cover which is not the union of m exact covers. Recently Ming-Zhi Zhang proved in [6] that for each $m=2,3,4,\ldots$ there exists an exactly m times cover no subcover of which is an exactly $n<m$ times cover.

Despite the negative answer to Porubský's question, we give here a result which has a positive aspect in some sense.

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THEOREM: *Let (1) be an exactly m times cover. Then for each $n = 1,\ldots,m$ there exist (at least) $\binom{m}{n}$ subsets I of $\{1,\ldots,k\}$ such that $\sum_{i\in I} 1/n_i$ equals n, in particular there are (at least) m subsets $I \subseteq \{1,\ldots,k\}$ with the property $\sum_{i\in I} 1/n_i = 1$.*

To prove it we need a lemma.

LEMMA: *Provided that $\{a_t+n_t\mathbb{Z}\}_{t=1}^k$ is an exactly m times cover, for any integer x we have the identity*

$$\prod_{t=1}^{k}(1 - z^{N/n_t}e^{2\pi i(x+a_t)/n_t}) = (1 - z^N)^m$$

where $N = [n_1,\ldots,n_k]$ is the least common multiple of $n_1,\ldots,n_k$.

Proof: Notice that any zero θ of the left side satisfies $\theta^N = 1$. Furthermore, for each $u \in \{0,1,\ldots,N-1\}$, $e^{2\pi i u/N}$ is a zero of multiplicity m of the left hand side because $-u-x$ is covered by $\{a_t + n_t\mathbb{Z}\}_{t=1}^k$ exactly m times. ∎

We remark that both the above lemma and the main result of J. Beebee [1] are easy consequences of a more general theorem (cf. Theorem 3 of Sun [4] and Theorem 4 of Sun [5]).

Proof of Theorem: Suppose $r \geq 0$. Letting $z = r^{1/[n_1,\ldots,n_k]}$ we obtain from the lemma

$$\prod_{t=1}^{k}(1 - r^{1/n_t}e^{2\pi i(n+a_t)/n_t}) = (1-r)^m, \quad n = 1,2,3,\ldots.$$

Hence for all $n \in \mathbb{Z}^+$ we have

$$1-\sum_{t=1}^{k} r^{1/n_t}e^{2\pi i(n+a_t)/n_t} + \sum_{1\leq t_1<t_2\leq k} r^{1/n_{t_1}+1/n_{t_2}}e^{2\pi i((n+a_{t_1})/n_{t_1}+(n+a_{t_2})/n_{t_2})} - \cdots + (-1)^k r^{1/n_1+\cdots+1/n_k}e^{2\pi i((n+a_1)/n_1+\cdots+(n+a_k)/n_k)} = (1-r)^m.$$

For $s > 1$, $\sum_{n=1}^{\infty} e^{2\pi i c n}/n^s$ converges absolutely, and by the above

$$\sum_{\emptyset\subset I\subseteq\{1,\ldots,k\}} (-1)^{|I|} r^{\sum_{t\in I}1/n_t} e^{2\pi i\sum_{t\in I}a_t/n_t} \sum_{n=1}^{\infty}\frac{e^{2\pi i\sum_{t\in I}n/n_t}}{n^s} = \sum_{n=1}^{\infty}\frac{1}{n^s}[(1-r)^m - 1],$$

i.e.

$$\left[\sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t\in\mathbb{Z}}}(-1)^{|I|}r^{\sum_{t\in I}1/n_t}e^{2\pi i\sum_{t\in I}a_t/n_t}+1-(1-r)^m\right]\zeta(s)$$
$$+\sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t\notin\mathbb{Z}}}(-1)^{|I|}r^{\sum_{t\in I}1/n_t}e^{2\pi i\sum_{t\in I}a_t/n_t}\sum_{n=1}^{\infty}\frac{e^{2\pi in\sum_{t\in I}1/n_t}}{n^s}=0.$$

Let $s\to 1$ from the right. Then $\zeta(s)\to\infty$, and if $\sum_{t\in I}1/n_t\notin\mathbb{Z}$ then $\sum_{n=1}^{\infty}e^{2\pi in\sum_{t\in I}1/n_t}/n^s$ has a finite limit. From the equality we must have

$$1+\sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t\in\mathbb{Z}}}(-1)^{|I|}r^{\sum_{t\in I}1/n_t}e^{2\pi i\sum_{t\in I}a_t/n_t}=(1-r)^m.$$

The last equality holds for any $r\geq 0$, and by comparing the coefficients we get

$$(2)\qquad \sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t=n}}(-1)^{|I|}e^{2\pi i\sum_{t\in I}a_t/n_t}=(-1)^n\binom{m}{n}\quad\text{for } n=1,\dots,m.$$

Given $n\in\{1,\dots,m\}$ we have

$$\binom{m}{n}=\left|\sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t=n}}(-1)^{|I|}e^{2\pi i\sum_{t\in I}a_t/n_t}\right|\leq\sum_{\substack{\emptyset\subset I\subseteq\{1,\dots,k\}\\ \sum_{t\in I}1/n_t=n}}1,$$

so there are at least $\binom{m}{n}$ subsets I of $\{1,\dots,k\}$ such that $\sum_{t\in I}1/n_t$ equals n. This concludes the proof. ∎

Observe that m sequences $\mathbb{Z}$ form an exactly m times cover. This example shows that the lower bounds $\binom{m}{n}$ $(1\leq n\leq m)$ are best possible.

At the end we mention that in 1989 Ming-Zhi Zhang [7] obtained the following surprising result: If $\bigcup_{i=1}^{k}a_i+n_i\mathbb{Z}=\mathbb{Z}$ then $\sum_{i\in I}1/n_i\in\mathbb{Z}^+$ for some $I\subseteq\{1,\dots,k\}$.

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